

RATE OF CONVERGENCE OF ATTRACTORS FOR SEMILINEAR SINGULARLY PERTURBED PROBLEMS: SCALAR PARABOLIC EQUATIONS WITH LOCALIZED LARGE DIFFUSION

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ABSTRACT. In this paper we study the asymptotic nonlinear dynamics of scalar semilinear parabolic problems reaction-diffusion type when the diffusion coefficient becomes large in a subregion which is interior to the domain. We obtain, under suitable assumptions, that the family of attractors behaves continuously and we exhibit the rate of convergence. An accurate description of localized large diffusion is necessary.

1. INTRODUCTION

Local spatial homogenization is a natural feature that appears in several physical phenomena, it is often present in heat conduction in composite materials for which the heat may be conducted much more quickly in some regions than in others and in reaction-diffusion problems for which the diffusivity vary considerably from one region to another have solutions that tend to become spatially homogeneous where the diffusivity is large. There are many studies available for mathematical models with such properties (see, for example, [4, 6] and [15]). In [10] the authors considered a scalar parabolic problem where the diffusivity is large except in a neighborhood of a finite number of points where it becomes small (see also, [12]). There it was shown that the asymptotic behavior is described by a system of linearly coupled ordinary differential equations. The analysis in [10] requires a detailed description, of the transition between large and small, of the diffusivity.

In this paper we discuss scalar parabolic problems with localized large diffusion, that is, in the situation when the diffusivity

in the viewpoint of rate of convergence of attractors where we will make an accurate description of localized large diffusion. To better present the main ideas while avoiding excessive notation, we consider the case where the diffusion is large only in a part of the domain, leaving the case where the diffusion is large in a finite number of part of the domain implicit.

Consider the scalar parabolic problem

$$(1) \quad \begin{cases} u_t^\varepsilon - (p_\varepsilon(x)u_x^\varepsilon)_x + (\lambda + c(x))u^\varepsilon = f(u^\varepsilon), & 0 < x < 1, \ t > 0, \\ u_x^\varepsilon(0) = u_x^\varepsilon(1) = 0, & t > 0, \\ u^\varepsilon(0) = u_0^\varepsilon, \end{cases}$$

where $\varepsilon \in (0, \varepsilon_0]$ is a parameter ($0 < \varepsilon_0 < 1$), $c \in C^1([0, 1])$, $f \in C^2(\mathbb{R})$ and $\lambda \in \mathbb{R}$ is such that $0 < m_0 \leq \min_{x \in [0, 1]} c(x) + \lambda$, for some positive constant m_0 . To describe the coefficients p_ε , let $0 = x_0 < x_1 < x_2 < x_3 = 1$ be a partition of the interval $\Omega = (0, 1)$. We assume the diffusion is very large in the open interval $\Omega_0 = (x_1, x_2)$ and converges uniformly to $p_0 \in C^2(\Omega_1)$ in the $\Omega_1 = [0, x_1] \cup [x_2, 1]$ as ε approaches to zero. More precisely, for $\varepsilon \in (0, \varepsilon_0]$, $p_\varepsilon \in C^2([0, 1])$ satisfies the following conditions (see Fig 1).

$$(2) \quad \begin{cases} p_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} p_0, & \text{uniformly in } \Omega_1, \\ p_\varepsilon(x) \geq \frac{1}{\varepsilon} & \text{in } [x_1 + \varepsilon, x_2 - \varepsilon], \\ m_0 \leq p_\varepsilon & \text{in } \Omega \quad \text{and} \quad m_0 \leq p_0 \quad \text{in } \Omega_1. \end{cases}$$

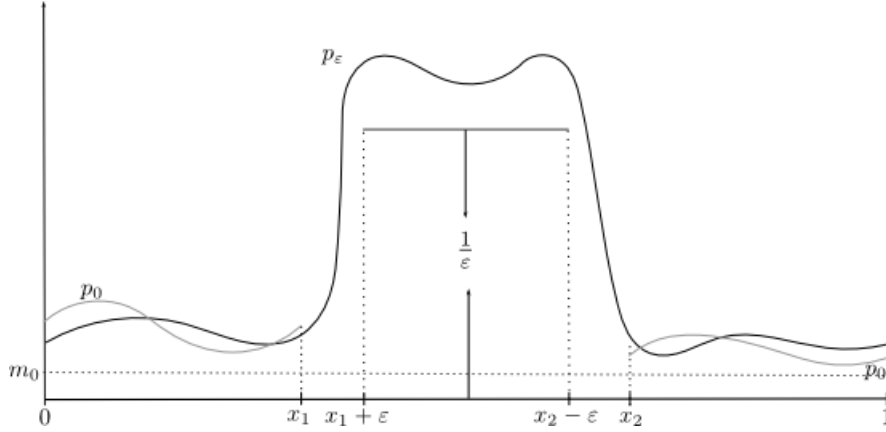


FIGURE 1. Diffusion

Due to the large diffusion it is natural to expect that the solutions u^ε of the (1) become approximately spatially constant on Ω_0 . Thus by assuming that the solutions u^ε exist and converge as $\varepsilon \rightarrow 0$, in some sense, for a function $u^0(t, x)$ which is spatially constant on (x_1, x_2) that we denote $u^0(t, x) = u_{\Omega_0}^0(t)$ for all $x \in \Omega_0$, the limiting problem of (1) as $\varepsilon \rightarrow 0$ is given by

$$(3) \quad \begin{cases} u_t^0 - (p_0(x)u_x^0)_x + (\lambda + c(x))u^0 = f(u^0), & x \in \Omega_1, t > 0, \\ u_x^0(0) = u_x^0(1) = 0, & t > 0, \\ u^0|_{\Omega_0} = u_{\Omega_0}^0, \\ \dot{u}_{\Omega_0}^0 + (\lambda + c_{\Omega_0})u_{\Omega_0}^0 = f(u_{\Omega_0}^0), & x \in \Omega_0, \\ u^0(0) = u_0^0, \end{cases}$$

where $c_{\Omega_0} = \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} c dx$ and $u_0^0 = \lim_{\varepsilon \rightarrow 0} u_0^\varepsilon$ which are constant on Ω_0 . In [15] it was formally proved that (3) determines the asymptotic behavior of (1).

In order to write (1) and (3) abstractly in a suitable natural energy space we introduce some more terminology. We define the operator $A_\varepsilon : \mathcal{D}(A_\varepsilon) \subset L^2(0, 1) \rightarrow L^2(0, 1)$ by

$$\mathcal{D}(A_\varepsilon) = \{u \in H^2(0, 1); u_x(0) = u_x(1) = 0\} \quad \text{and} \quad A_\varepsilon u = -(p_\varepsilon u_x)_x + (\lambda + c)u.$$

We denote

$$L_{\Omega_0}^2(0, 1) = \{u \in L^2(0, 1); u \text{ is constant a.e. in } \Omega_0\},$$

$$H_{\Omega_0}^1(0, 1) = \{u \in H^1(0, 1); u_x = 0 \text{ in } \Omega_0\}$$

and we define the operator $A_0 : \mathcal{D}(A_0) \subset L_{\Omega_0}^2(0, 1) \rightarrow L_{\Omega_0}^2(0, 1)$ by

$$\mathcal{D}(A_0) = \{u \in H_{\Omega_0}^1(0, 1); -(p_0 u_x)_x \in L^2(\Omega_1), u_x(0) = u_x(1) = 0\};$$

$$A_0 u = [- (p_0 u_x)_x + (\lambda + c)u] \chi_{\Omega_1} + [(\lambda + c_{\Omega_0})u] \chi_{\Omega_0}.$$

It is well known that A_ε is a self-adjoint invertible operator with compact resolvent for each $\varepsilon \in [0, \varepsilon_0]$. Moreover, if λ_0^ε is the first eigenvalue of A_ε and $\xi > \lambda_0^\varepsilon$, then the operator $A_\varepsilon - \xi I$ is positive and hence we can define, in the usual way (see [13]), the fractional power space $X_\varepsilon^{\frac{1}{2}} = H^1(0, 1)$, $\varepsilon \in (0, \varepsilon_0]$, and $X_0^{\frac{1}{2}} = H_{\Omega_0}^1(0, 1)$ with the scalar products

$$\langle u, v \rangle_{X_\varepsilon^{\frac{1}{2}}} = \int_0^1 p_\varepsilon u_x v_x dx + \int_0^1 (\lambda + c)uv dx, \quad u, v \in X_\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, \varepsilon_0];$$

$$\langle u, v \rangle_{X_0^{\frac{1}{2}}} = \int_{\Omega_1} p_0 u_x v_x dx + \int_0^1 (\lambda + c)uv dx, \quad u, v \in X_0^{\frac{1}{2}}.$$

The space $X_0^{\frac{1}{2}}$ is a closed subspace of $X_\varepsilon^{\frac{1}{2}}$, $\varepsilon \in (0, \varepsilon_0]$ and $X_\varepsilon^{\frac{1}{2}} \subset H^1(0, 1)$ with injection constant independent of ε . However a delicate issue here is that the injection $H^1(0, 1) \subset X_\varepsilon^{\frac{1}{2}}$ has not constant independent of ε , indeed is valid

$$\|u\|_{H^1} \leq C \|u\|_{X_\varepsilon^{\frac{1}{2}}} \leq M_\varepsilon \|u\|_{H^1}, \quad u \in H^1,$$

where $M_\varepsilon \rightarrow \infty$ as $\varepsilon \rightarrow 0$. Actually, it can be proved that there is no possible uniform choice for M_ε (see [11]), therefore estimates in the space H^1 does not guarantee estimates in $X_\varepsilon^{\frac{1}{2}}$. We will consider $X_\varepsilon^{\frac{1}{2}}$ as the phase space for the problems (1) and (3) that is, if we denote the Nemitskii functional of f by the same notation f , then (1) and (3) can be written as

$$(4) \quad \begin{cases} u_t^\varepsilon + A_\varepsilon u^\varepsilon = f(u^\varepsilon), \\ u^\varepsilon(0) = u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in [0, \varepsilon_0]. \end{cases}$$

Since $X_\varepsilon^{\frac{1}{2}} \subset C([0, 1])$, if we assume that the nonlinearity f satisfies the following dissipativeness condition

$$\limsup_{|x| \rightarrow \infty} \frac{f(x)}{x} < 0.$$

It follows from [4, 2] and [3] that we can consider f bounded globally Lipschitz and the problem (4), for each $\varepsilon \in [0, \varepsilon_0]$, is well posed for positive time and the solutions are continuously differentiable with respect to the initial data. Thus we are able to consider in $X_\varepsilon^{\frac{1}{2}}$ the family of nonlinear semigroups $\{T_\varepsilon(\cdot)\}_{\varepsilon \in [0, \varepsilon_0]}$ defined by $T_\varepsilon(t) = u^\varepsilon(t, u_0^\varepsilon)$, $t \geq 0$, where $u^\varepsilon(t, u_0^\varepsilon)$ is the solution of (4) through $u_0^\varepsilon \in X_\varepsilon^{\frac{1}{2}}$ and

$$(5) \quad T_\varepsilon(t)u_0^\varepsilon = e^{-A_\varepsilon t}u_0^\varepsilon + \int_0^t e^{-A_\varepsilon(t-s)}f(T_\varepsilon(s))ds, \quad t \geq 0,$$

has a global attractor \mathcal{A}_ε , for each $\varepsilon \in [0, \varepsilon_0]$ such that $\overline{\bigcup_{\varepsilon \in [0, \varepsilon_0]} \mathcal{A}_\varepsilon}$ is compact and uniformly bounded.

We recall that $\{\mathcal{A}_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$ is continuous at $\varepsilon = 0$ if

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) = \text{dist}_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) + \text{dist}_H(\mathcal{A}_0, \mathcal{A}_\varepsilon) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$\text{dist}_H(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|_{X_\varepsilon^{\frac{1}{2}}}, \quad A, B \subset X_\varepsilon^{\frac{1}{2}}.$$

We also recall that the equilibria solutions of (4) are those which are independent of time, that is, for $\varepsilon \in [0, \varepsilon_0]$, they are the solutions of the elliptic problem $A_\varepsilon u^\varepsilon - f(u^\varepsilon) = 0$. We denote by \mathcal{E}_ε the set of the equilibria solutions of A_ε and we say that $u_*^\varepsilon \in \mathcal{E}_\varepsilon$ is an hyperbolic solution if $\sigma(A_\varepsilon - f'(u_*^\varepsilon)) \cap \{\mu \in \mathbb{C}; \text{Re}(\mu) = 0\} = \emptyset$. We assume \mathcal{E}_0 is composed of hyperbolic solutions, then \mathcal{E}_0 is finite and the family $\{\mathcal{E}_\varepsilon\}_{\varepsilon \in [0, \varepsilon_0]}$ is continuous at $\varepsilon = 0$ (see [6]), thus for ε sufficiently small \mathcal{E}_ε is composed of a finite number of hyperbolic solutions and the semigroups in (5) are dynamically gradient with respect to \mathcal{E}_ε , for each $\varepsilon \in [0, \varepsilon_0]$. Moreover in [7] the authors showed that the semigroup $T_0(\cdot)$ is Morse-Smale and the main result for Morse-Smale semigroups is stability of the phase diagram under perturbation (see [5]). In [7] the authors also proved the gap condition for eigenvalues of the operators A_ε , $\varepsilon \in [0, \varepsilon_0]$, and then the existence of exponential attracting finite dimensional inertial manifolds \mathcal{M}_ε containing \mathcal{A}_ε is ensured. Thus we can restrict the semigroups T_ε to these inertial manifolds in order to obtain a finite dimensional problem where the robustness these inertial manifolds also ensure the geometric and topological equivalence of the attractors.

Under these assumptions, the authors in [4] and [6] proved the continuity of attractors of the problem (4) in the phase space $X_\varepsilon^{\frac{1}{2}}$ however rate of attraction was not considered. The main result about rate of attraction of Morse-Smale problem is due to [16]. They obtained an almost optimal rate of convergence of attractors involving a compact convergence of the resolvent operators $\|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})}$. Following these ideas in this paper we will exhibit a rate of convergence for the continuity of attractors of the problem (4) in the phase space $X_\varepsilon^{\frac{1}{2}}$ depending on the diffusion coefficients p_ε and the parameter ε .

This paper is organized as follows. In the section 2 we make the study of the elliptic problem in order to find a rate of attraction for resolvent operator. In the section 3 we exhibit a rate of attraction for the eigenvalues and equilibrium points. In section 4 we obtain the rate of convergence of the invariant manifold and in the Section 5 we reduce the system to finite dimensional and we finally obtain a rate of convergence of attractors.

2. ELLIPTIC PROBLEM

In this section we analyze the solvability of the elliptic problem associated to (4) in order to obtain the rate of convergence of the resolvent operators. As a consequence we will estimate the time one map $T_\varepsilon(1)$ associated to (5). Later we will transfer such estimate for the time one map restricted to finite dimensional invariant manifold.

Next result establishes the convergence of the resolvent operator $A_\varepsilon^{-1}|_{L^2_{\Omega_0}}$ to A_0^{-1} and ensures that the rate of this convergence is $(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}$. Since we have a singular perturbation we can consider the nonlinearity on the limit space.

Lemma 2.1. For $g \in L^2_{\Omega_0}$ with $\|g\|_{L^2} \leq 1$ and $\varepsilon \in [0, \varepsilon_0]$, let u^ε be the solution of elliptic problem

$$(6) \quad \begin{cases} -(p_\varepsilon(x)u_x^\varepsilon)_x + (\lambda + c(x))u^\varepsilon = g, & x \in (0, 1), \\ u_x^\varepsilon(0) = u_x^\varepsilon(1) = 0. \end{cases}$$

Then there is a constant $C > 0$ independent of ε such that

$$(7) \quad \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

Proof. First note that the weak solution u^ε satisfies

$$(8) \quad \int_0^1 p_\varepsilon u_x^\varepsilon \varphi_x dx + \int_0^1 (\lambda + c)u^\varepsilon \varphi dx = \int_0^1 g \varphi dx, \quad \forall \varphi \in X_\varepsilon^{\frac{1}{2}}, \quad \varepsilon \in (0, \varepsilon_0];$$

$$(9) \quad \int_{\Omega_1} p_0 u_x^0 \varphi_x dx + \int_0^1 (\lambda + c)u^0 \varphi dx = \int_0^1 g \varphi dx, \quad \forall \varphi \in X_0^{\frac{1}{2}}.$$

If we take $\varphi = u^\varepsilon$ as a test function we get reach uniform bound for weak solution u^ε in the spaces H^1 and $X_\varepsilon^{\frac{1}{2}}$ for $\varepsilon \in [0, \varepsilon_0]$. Also the embedding $H^1 \subset L^\infty$ gives us an uniform bound for u^ε in the space L^∞ . Moreover we have $\|u_x^\varepsilon\|_{L^\infty} \leq m_0^{-1}(1 + \lambda + \max_{x \in [0, 1]} c(x))$.

In what follows we denote C for any positive constant independent of ε .

We define the linear operator $E : X_\varepsilon^{\frac{1}{2}} \rightarrow X_0^{\frac{1}{2}}$ by

$$Eu = \begin{cases} u & \text{in } [0, x_1 - \varepsilon] \cup [x_2 + \varepsilon, 1], \\ \text{linear} & \text{in } [x_1 - \varepsilon, x_1] \cup [x_2, x_2 + \varepsilon], \\ \bar{u} := \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} u \, dx & \text{in } \Omega_0, \end{cases}$$

for all $u \in X_\varepsilon^{\frac{1}{2}}$. We denote $\Omega_\varepsilon = [x_1 - \varepsilon, x_1] \cup [x_2, x_2 + \varepsilon]$. and if we let $u^\varepsilon - u^0$ as a test function in (8) and if we let $E(u^\varepsilon - u^0)$ as a test function in (9), we have

$$\begin{aligned} \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}}^2 &\leq \int_{\Omega_\varepsilon} g |u^\varepsilon - u^0| \, dx + \int_{\Omega_\varepsilon} g |E(u^\varepsilon - u^0)| \, dx + \int_{\Omega_1} |p_\varepsilon - p_0| |u_x^0| |u_x^\varepsilon - u_x^0| \, dx \\ &+ \int_{\Omega_\varepsilon} p_\varepsilon |u_x^\varepsilon - u_x^0|^2 \, dx + \int_{\Omega_\varepsilon} p_\varepsilon |u_x^\varepsilon| |u_x^\varepsilon - u_x^0| \, dx + \int_{\Omega_\varepsilon} p_0 |u_x^0| |E(u^\varepsilon - u^0)_x| \, dx \\ &+ \int_{\Omega_\varepsilon} (\lambda + c) |u^\varepsilon| |u^\varepsilon - u^0| \, dx + \int_{\Omega_\varepsilon} (\lambda + c) |u^0| |E(u^\varepsilon - u^0)| \, dx + \int_{\Omega_\varepsilon} (\lambda + c) |u^\varepsilon - u^0|^2 \, dx. \end{aligned}$$

Since p_ε converges uniformly to p_0 in Ω_1 , we have p_ε uniformly bounded in Ω_1 . Thus by Hölder inequality, (2) and the uniform bound for weak solution u^ε and u_x^ε , each integrals on the right hand side of the above expression can be estimated by $C \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}} (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon^{\frac{1}{2}})$.

For terms with evolve the operator E we have used

$$\int_{\Omega_\varepsilon} |E(u^\varepsilon - u^0)_x| \, dx \leq C \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}} \varepsilon^{\frac{1}{2}} \text{ and } \|E(u^\varepsilon - u^0)\|_{L^\infty(\Omega_\varepsilon)} \leq C \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}}.$$

We will prove the first one. For this we denote $v^\varepsilon = u^\varepsilon - u^0$ and, just to short, we assume $\Omega_\varepsilon = [x_1 - \varepsilon, x_1]$. In this case we have

$$|E(v^\varepsilon)_x| = \left| \frac{\bar{v}^\varepsilon - v^\varepsilon(x_1 - \varepsilon)}{\varepsilon} \right| \leq \left| \frac{\bar{v}^\varepsilon - v^\varepsilon(x_1 + \varepsilon)}{\varepsilon} \right| + \left| \frac{v^\varepsilon(x_1 + \varepsilon) - v^\varepsilon(x_1 - \varepsilon)}{\varepsilon} \right|,$$

thus

$$\begin{aligned} |\bar{v}^\varepsilon - v^\varepsilon(x_1 + \varepsilon)| &\leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} |v^\varepsilon - v^\varepsilon(x_1 + \varepsilon)| \, dx \\ &= \frac{1}{x_2 - x_1} \left[\int_{x_1}^{x_1 + \varepsilon} + \int_{x_1 + \varepsilon}^{x_2 - \varepsilon} + \int_{x_2 - \varepsilon}^{x_2} |v^\varepsilon - v^\varepsilon(x_1 + \varepsilon)| \, dx \right]. \end{aligned}$$

We have

$$\int_{x_1}^{x_1 + \varepsilon} + \int_{x_2 - \varepsilon}^{x_2} |v^\varepsilon - v^\varepsilon(x_1 + \varepsilon)| \, dx \leq C \|v^\varepsilon\|_{L^\infty} \varepsilon \leq C \|v^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} \varepsilon^{\frac{1}{2}}$$

and for $x \in [x_1 + \varepsilon, x_2 - \varepsilon]$,

$$|v^\varepsilon(x) - v^\varepsilon(x_1 + \varepsilon)| \leq \int_{x_1 + \varepsilon}^{x_2 - \varepsilon} |v_x^\varepsilon| \, dx \leq \left(\int_{x_1 + \varepsilon}^{x_2 - \varepsilon} |v_x^\varepsilon|^2 \, dx \right)^{\frac{1}{2}},$$

but

$$\frac{1}{\varepsilon} \int_{x_1 + \varepsilon}^{x_2 - \varepsilon} |v_x^\varepsilon|^2 \, dx \leq \int_{x_1 + \varepsilon}^{x_2 - \varepsilon} p_\varepsilon |v_x^\varepsilon|^2 \, dx \leq \|v^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}}^2$$

and then

$$\int_{x_1+\varepsilon}^{x_2-\varepsilon} |v^\varepsilon(x) - v^\varepsilon(x_1 + \varepsilon)| dx \leq C \|v^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} \varepsilon^{\frac{1}{2}}.$$

We also have

$$|v^\varepsilon(x_1 + \varepsilon) - v^\varepsilon(x_1 - \varepsilon)| \leq \int_{x_1-\varepsilon}^{x_1+\varepsilon} |v_x^\varepsilon| dx \leq \left(\int_{x_1-\varepsilon}^{x_1+\varepsilon} |v_x^\varepsilon|^2 dx \right)^{\frac{1}{2}} (2\varepsilon)^{\frac{1}{2}} \leq C \|v^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} \varepsilon^{\frac{1}{2}}.$$

□

As a consequence of the previous result, we have the following result.

Corollary 2.2. There is a positive constant C independent of ε such that

$$(10) \quad \|A_\varepsilon^{-1} - A_0^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

Furthermore, there is $\phi \in (\frac{\pi}{2}, \pi)$ such that for all $\mu \in \Sigma_{\bar{\lambda}, \phi} = \{\mu \in \mathbb{C} : |\arg(\mu + \bar{\mu})| \leq \phi\} \setminus \{\mu \in \mathbb{C} : |\mu + \bar{\lambda}| \leq r\}$, where $\bar{\lambda} = \lambda + c_{\Omega_0}$ and $r > 0$.

$$(11) \quad \|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

Proof. The first part is an immediate consequence of Lemma 2.1. Let $\rho(A_\varepsilon)$ be the resolvent set of the operator A_ε , $\varepsilon \in [0, \varepsilon_0]$. If $\mu \in \rho(-A_\varepsilon) \cap \rho(-A_0)$, we choose $\varphi \in (\frac{\pi}{2}, \pi)$ suitable in order to get the sectorial estimates

$$\|(\mu + A_\varepsilon)^{-1}\|_{\mathcal{L}(L^2)} \leq \frac{M_\varphi}{|\mu|}, \quad \varepsilon \in (0, \varepsilon_0], \quad \text{and} \quad \|(\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2)} \leq \frac{M_\varphi}{|\mu|}.$$

Therefore, $\|A_\varepsilon(\mu + A_\varepsilon)^{-1}\|_{\mathcal{L}(L^2)} \leq 1 + M_\varphi$ for $\varepsilon \in (0, \varepsilon_0]$ and $\|A_0(\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2)} \leq 1 + M_\varphi$.

But if $g \in L_{\Omega_0}^2$, we can write

$$A_\varepsilon^{\frac{1}{2}} \left((\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1} \right) g = A_\varepsilon(\mu + A_\varepsilon)^{-1} A_\varepsilon^{\frac{1}{2}} (A_\varepsilon^{-1} - A_0^{-1}) A_0(\mu + A_0)^{-1} g$$

and thus

$$\begin{aligned} & \|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \\ & \leq \|A_\varepsilon(\mu + A_\varepsilon)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2)} \|A_\varepsilon^{\frac{1}{2}} (A_\varepsilon^{-1} - A_0^{-1})\|_{\mathcal{L}(L_{\Omega_0}^2)} \|A_0(\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2)} \\ & \leq \|A_\varepsilon(\mu + A_\varepsilon)^{-1}\|_{\mathcal{L}(L^2)} \|A_\varepsilon^{\frac{1}{2}} (A_\varepsilon^{-1} - A_0^{-1})\|_{\mathcal{L}(L_{\Omega_0}^2)} \|A_0(\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2)} \\ & \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}, \end{aligned}$$

for some constant $C = C(\varphi) > 0$ independent of μ and ε . □

In the next theorem we will obtain the rate of convergence of nonlinear semigroup. We will follow [16] that improves (for the Morse-Smale case) the results presented by [1]. For our purposes we just need to consider the time $t = 1$.

Theorem 2.3. For each $w_0 \in \mathcal{A}_0$, there is a positive constant C independent of ε such that

$$\|T_\varepsilon(1)w_0 - T_0(1)w_0\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} |\log(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)|.$$

Proof. For each $\varepsilon \in [0, \varepsilon_0]$ the operator A_ε generates an analytic semigroup $\{e^{-A_\varepsilon t}; t \geq 0\}$ which is given by

$$e^{-A_\varepsilon t} = \frac{1}{2\pi i} \int_\Gamma e^{\mu t} (\mu + A_\varepsilon)^{-1} d\mu,$$

where Γ is the boundary of $\Sigma_{-\alpha, v} \setminus \{\mu \in \mathbb{C}; |\mu + \alpha| \leq r\}$ for some small $r, v \in (\phi, \frac{\pi}{2})$ and $\alpha > 0$, oriented towards increasing of the imaginary part with $\Sigma_{-\alpha, v} = \{\mu \in \mathbb{C}; |\arg(\mu + \alpha)| \leq v\}$. It follows that there is $0 < \gamma = \gamma(\varepsilon) \rightarrow \infty$ as $\varepsilon \rightarrow 0$ such that,

$$(12) \quad \|e^{-A_\varepsilon t}\|_{\mathcal{L}(L^2, X_\varepsilon^{\frac{1}{2}})} \leq M t^{-\frac{1}{2}} e^{-\gamma t}, \quad t > 0, \quad \varepsilon \in (0, \varepsilon_0];$$

$$(13) \quad \|e^{-A_0 t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq M t^{-\frac{1}{2}} e^{-\gamma t}, \quad t > 0,$$

where M is a positive constant independent of ε . Thus for $t > 0$,

$$\begin{aligned} \|e^{-A_\varepsilon t} - e^{-A_0 t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} &\leq \|e^{-A_\varepsilon t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} + \|e^{-A_0 t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \\ &\leq M t^{-\frac{1}{2}} e^{-\gamma t} + M t^{-\frac{1}{2}} e^{-\gamma t} \\ &\leq 2M t^{-\frac{1}{2}}. \end{aligned}$$

Moreover, using the Theorem 2.2,

$$\begin{aligned} \|e^{-A_\varepsilon t} - e^{-A_0 t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} &\leq \frac{1}{2\pi} \int_\Gamma |e^{\mu t}| \|(\mu + A_\varepsilon)^{-1} - (\mu + A_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} |d\mu| \\ &\leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} t^{-1}. \end{aligned}$$

We denote $l_\varepsilon(t) = \max\{t^{-\frac{1}{2}}, (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} t^{-1}\}$. Since the nonlinear semigroup is given by (5), then for $0 < t \leq 1$, we have

$$\begin{aligned} \|T_\varepsilon(t)w_0 - T_0(t)w_0\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \|(e^{-A_\varepsilon t} - e^{-A_0 t})w_0\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \int_0^t \|e^{-A_\varepsilon(t-s)} f(T_\varepsilon(s)w_0) - e^{-A_0(t-s)} f(T_0(s)w_0)\|_{X_\varepsilon^{\frac{1}{2}}} ds, \end{aligned}$$

but

$$\begin{aligned}
& \int_0^t \|e^{-A_\varepsilon(t-s)} f(T_\varepsilon(s)w_0) - e^{-A_0(t-s)} f(T_0(s)w_0)\|_{X_\varepsilon^{\frac{1}{2}}} ds \\
& \leq \int_0^t \|e^{-A_\varepsilon(t-s)} [f(T_\varepsilon(s)w_0) - f(T_0(s)w_0)]\|_{X_\varepsilon^{\frac{1}{2}}} ds \\
& \quad + \int_0^t \|e^{-A_\varepsilon(t-s)} f(T_0(s)w_0) - e^{-A_0(t-s)} f(T_0(s)w_0)\|_{X_\varepsilon^{\frac{1}{2}}} ds \\
& \leq C \int_0^t (t-s)^{-\frac{1}{2}} \|T_\varepsilon(s)w_0 - T_0(s)w_0\|_{X_\varepsilon^{\frac{1}{2}}} ds \\
& \quad + C \int_0^t l_\varepsilon(t-s) \|f(T_0(s)w_0)\|_{X_\varepsilon^{\frac{1}{2}}} ds
\end{aligned}$$

If we denote $\varphi(t) = \|T_\varepsilon(t)w_0 - T_0(t)w_0\|_{X_\varepsilon^{\frac{1}{2}}} t^{\frac{1}{2}}$ and $\tau(\varepsilon) = (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}$, we have

$$\varphi(t) \leq [C\tau(\varepsilon) + C|\log(\tau(\varepsilon))\tau(\varepsilon)|]t^{-\frac{1}{2}} + C \int_0^t (t-s)^{-\frac{1}{2}} \varphi(s) ds,$$

where we have used the the following estimate (see Lemma 3.10 in [16]),

$$\int_0^\infty e^{-\gamma(t-s)} l_\varepsilon(t-s) ds \leq 4 \log(\tau(\varepsilon)) \tau(\varepsilon).$$

The result follows by Gronwall's inequality (see Lemma 6.25 in [9]), taking $t = 1$. \square

3. RATE OF CONVERGENCE OF EIGENVALUES AND EQUILIBRIA

In this section we will obtain the rate of convergence of eigenvalues, spectral projection and equilibrium points.

The convergence of eigenvalues and eigenfunctions of the linear operators was proved in [15] and the properties about the compact convergence of the spectral projections was studied in details in several works.

In the next result we will follow [1], where it was considered rate of convergence for the eigenvalues and spectral projections.

Lemma 3.1. If $\lambda^\varepsilon \in \sigma(A_\varepsilon)$, $\varepsilon \in [0, \varepsilon_0]$, and $\lambda^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \lambda^0$, then

$$|\lambda^\varepsilon - \lambda^0| \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

Moreover, if we denote $\sigma(A_\varepsilon) = \{\lambda_i^\varepsilon\}_{i=0}^\infty$ (ordered and counting multiplicity), we have the following gap condition

$$\lambda_{i+1}^\varepsilon - \lambda_i^\varepsilon \xrightarrow{i \rightarrow \infty} \infty.$$

Proof. Let $\lambda^0 \in \sigma(A_0)$ be an isolated eigenvalue. We consider an appropriated closed curve Γ in $\rho(-A_0)$ around λ^0 and define the spectral projection

$$Q_\varepsilon = \frac{1}{2\pi i} \int_\Gamma (\mu + A_\varepsilon)^{-1} d\mu, \quad \varepsilon \in [0, \varepsilon_0].$$

It follows from Corollary 2.2 that

$$\|Q_\varepsilon - Q_0\|_{\mathcal{L}(L^2_{\Omega_0}, X_\varepsilon^{\frac{1}{2}})} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

If we have $\lambda^\varepsilon \in \sigma(A_\varepsilon)$ such that $\lambda^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} \lambda^0$ then for ε sufficiently small there is $u^0 \in \text{Ker}(\lambda_0 - A_0)$ with $\|u^0\|_{X_\varepsilon^{\frac{1}{2}}} = 1$ such that $Q_\varepsilon u^0$ is eigenvalue of A_ε associated with λ^ε , thus

$$|\lambda^\varepsilon - \lambda^0| \leq \|(\lambda^\varepsilon - \lambda^0)u^0\|_{X_\varepsilon^{\frac{1}{2}}} = \|\lambda^\varepsilon Q_0 u^0 - \lambda^0 u^0\|_{X_\varepsilon^{\frac{1}{2}}},$$

but

$$\begin{aligned} \|\lambda^\varepsilon Q_0 u^0 - \lambda^0 u^0\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \|\lambda^\varepsilon Q_0 u^0 - \lambda^0 Q_\varepsilon u^0 + \lambda^0 Q_\varepsilon u^0 - \lambda^0 u^0\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq |\lambda^\varepsilon \lambda^0| \left\| \frac{1}{\lambda^0} Q_0 u^0 - \frac{1}{\lambda^\varepsilon} Q_\varepsilon u^0 \right\|_{X_\varepsilon^{\frac{1}{2}}} + \|\lambda^0 (Q_\varepsilon - Q_0) u^0\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq |\lambda^\varepsilon \lambda^0| \|A_0^{-1} Q_0 u^0 - A_\varepsilon^{-1} Q_\varepsilon u^0\|_{X_\varepsilon^{\frac{1}{2}}} + |\lambda^0| \|(Q_\varepsilon - Q_0) u^0\|_{X_\varepsilon^{\frac{1}{2}}}, \end{aligned}$$

and

$$\begin{aligned} \|A_0^{-1} Q_0 u^0 - A_\varepsilon^{-1} Q_\varepsilon u^0\|_{X_\varepsilon^{\frac{1}{2}}} &= \|A_0^{-1} Q_0 u^0 - A_\varepsilon^{-1} Q_0 u^0 + A_\varepsilon^{-1} Q_0 u^0 - A_\varepsilon^{-1} Q_\varepsilon u^0\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq \|(A_0^{-1} - A_\varepsilon^{-1}) Q_0 u^0\|_{X_\varepsilon^{\frac{1}{2}}} + \|A_\varepsilon^{-1} (Q_0 - Q_\varepsilon) u^0\|_{X_\varepsilon^{\frac{1}{2}}}. \end{aligned}$$

Hence the estimate for $|\lambda^\varepsilon - \lambda^0|$ follows.

In [7] it was proved that the eigenvalues of $-A_\varepsilon$ has a gap condition by characterizing these eigenvalues, that is, if $\sigma(-A_\varepsilon) = \{\mu_i^\varepsilon\}_{i=0}^\infty$ then

$$\mu_i^\varepsilon = -\frac{1}{l^2} i^2 \pi^2 + o(i)$$

as $i \rightarrow \infty$, where $l = \int_0^1 p_\varepsilon(s)^{-\frac{1}{2}} ds$. Consequently $\lambda_i^\varepsilon - \lambda_{i+1}^\varepsilon \xrightarrow{i \rightarrow \infty} \infty$. \square

Recall that we denote \mathcal{E}_ε the set of the equilibria solutions of the A_ε and we assume that \mathcal{E}_0 is composed of hyperbolic solutions, thus for ε sufficiently small \mathcal{E}_ε is composed of finite number of hyperbolic solutions. The rate of convergence of equilibrium points can be obtained as follows.

Theorem 3.2. Let $u_*^0 \in \mathcal{E}_0$. Then for ε sufficiently small (we still denote $\varepsilon \in (0, \varepsilon_0]$), there is $\delta > 0$ such that the equation $A_\varepsilon u - f(u) = 0$ has only solution $u_*^\varepsilon \in \{u \in X_\varepsilon^{\frac{1}{2}}; \|u - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}} \leq \delta\}$. Moreover

$$(14) \quad \|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}.$$

Proof. The proof is the same as given in [6]. Here we just need to prove the estimates (14). We have u_*^ε and u_*^0 given by

$$u_*^0 = (A_0 + V_0)^{-1} [f(u_*^0) + V_0 u_*^0] \quad \text{and} \quad u_*^\varepsilon = (A_\varepsilon + V_0)^{-1} [f(u_*^\varepsilon) + V_0 u_*^\varepsilon],$$

where $V_0 = -f'(u_*^0)$. Thus

$$\begin{aligned} \|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \|(A_\varepsilon + V_0)^{-1}[f(u_*^\varepsilon) + V_0 u_*^\varepsilon] - (A_0 + V_0)^{-1}[f(u_*^0) + V_0 u_*^0]\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq \|(A_\varepsilon + V_0)^{-1}[f(u_*^\varepsilon) - f(u_*^0) + V_0(u_*^\varepsilon - u_*^0)]\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \|[(A_\varepsilon + V_0)^{-1} - (A_0 + V_0)^{-1}][f(u_*^0) + V_0 u_*^0]\|_{X_\varepsilon^{\frac{1}{2}}}. \end{aligned}$$

We can prove that

$$(A_\varepsilon + V_0)^{-1} - (A_0 + V_0)^{-1} = [I - (A_\varepsilon + V_0)^{-1}V_0](A_\varepsilon^{-1} - A_0^{-1})[I - V_0(A_0 + V_0)^{-1}],$$

which implies $\|[(A_\varepsilon + V_0)^{-1} - (A_0 + V_0)^{-1}][f(u_*^0) + V_0 u_*^0]\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}$.

Now we denote $z^\varepsilon = f(u_*^\varepsilon) - f(u_*^0) + V_0(u_*^\varepsilon - u_*^0)$. Since f is continuously differentiable, for all $\delta > 0$ there is ε sufficiently small such that $\|z^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} \leq \delta\|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}}$, thus

$$\|(A_\varepsilon + V_0)^{-1}z^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} \leq \delta\|(A_\varepsilon + V_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})}\|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}}.$$

We choose δ sufficiently small such that $\delta\|(A_\varepsilon + V_0)^{-1}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq \frac{1}{2}$, thus

$$\|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} + \frac{1}{2}\|u_*^\varepsilon - u_*^0\|_{X_\varepsilon^{\frac{1}{2}}}.$$

□

Corollary 3.3. The family $\{\mathcal{E}_\varepsilon\}_{\varepsilon \in (0, \varepsilon_0]}$ is continuous at $\varepsilon = 0$. Moreover if $\mathcal{E}_0 = \{u_*^{0,1}, \dots, u_*^{0,k}\}$ then for ε sufficiently small, $\mathcal{E}_\varepsilon = \{u_*^{\varepsilon,1}, \dots, u_*^{\varepsilon,k}\}$ and

$$\|u_*^{\varepsilon,i} - u_*^{0,i}\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}, \quad i = 1, \dots, k.$$

4. RATE OF CONVERGENCE OF INVARIANT MANIFOLDS

In this section we characterize the invariant manifolds \mathcal{M}_ε locally as a graph of a Lipschitz function, and we guarantee that \mathcal{M}_ε approaches to the invariant manifold \mathcal{M}_0 when the parameter ε goes to zero. This result will be fundamental to reduce the study of the asymptotic dynamics of the problem (4) to a finite dimension.

We will follow the works [7] and [8].

The spectrum of $-A_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$, ordered and counting multiplicity is given by

$$\dots - \lambda_m^\varepsilon < 0 < -\lambda_{m-1}^\varepsilon < \dots < -\lambda_0^\varepsilon$$

with $\{\varphi_i^\varepsilon\}_{i=0}^\infty$ the eigenfunctions related. We consider the spectral projection onto the space generated by the first m eigenvalues, that is, if Γ is an appropriated closed curve in $\rho(-A_0)$ around $\{-\lambda_0^0, \dots, -\lambda_{m-1}^0\}$, then

$$Q_\varepsilon = \frac{1}{2\pi i} \int_\Gamma (\mu + A_\varepsilon)^{-1} d\mu, \quad \varepsilon \in [0, \varepsilon_0].$$

We observe that Q_ε is a projection of finite rank and then there is an isomorphism from $Q_\varepsilon X_\varepsilon^{\frac{1}{2}} = \text{span}[\varphi_0^\varepsilon, \dots, \varphi_{m-1}^\varepsilon]$ onto \mathbb{R}^m . Thus we can decompose $X_\varepsilon^{\frac{1}{2}} = Y_\varepsilon \oplus Z_\varepsilon$, where $Y_\varepsilon = Q_\varepsilon X_\varepsilon^{\frac{1}{2}}$ and $Z_\varepsilon = (I - Q_\varepsilon)X_\varepsilon^{\frac{1}{2}}$ and we define $A_\varepsilon^+ = A_\varepsilon|_{Y_\varepsilon}$ and $A_\varepsilon^- = A_\varepsilon|_{Z_\varepsilon}$ $\varepsilon \in [0, \varepsilon_0]$. The following estimates are valid.

- (i) $\|e^{-A_\varepsilon^+ t} z\|_{X_\varepsilon^{\frac{1}{2}}} \leq M e^{-\beta t} \|z\|_{X_\varepsilon^{\frac{1}{2}}}, \quad t \leq 0, \quad z \in Y_\varepsilon,$
- (ii) $\|e^{-A_\varepsilon^- t} z\|_{X_\varepsilon^{\frac{1}{2}}} \leq M e^{-\gamma t} \|z\|_{X_\varepsilon^{\frac{1}{2}}}, \quad t > 0, \quad z \in Z_\varepsilon,$
- (iii) $\|e^{-A_\varepsilon^+ t} - e^{A_0^+ t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq M e^{-\beta t} (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}, \quad t \leq 0,$
- (iv) $\|e^{-A_\varepsilon^- t} - e^{A_0^- t}\|_{\mathcal{L}(L_{\Omega_0}^2, X_\varepsilon^{\frac{1}{2}})} \leq M e^{-\gamma t} l_\varepsilon(t), \quad t > 0.$

where $l_\varepsilon(t) = \max\{t^{-\frac{1}{2}}, (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} t^{-1}\}$, $\gamma = \gamma(\varepsilon) = \lambda_{m+1}^\varepsilon$, $\beta = \lambda_m^0 + 1$, and M is a positive constant independent of ε .

Theorem 4.1. For ε sufficiently small there is an invariant manifold \mathcal{M}_ε for (4) given by

$$\mathcal{M}_\varepsilon = \{u^\varepsilon \in X_\varepsilon^{\frac{1}{2}}; u^\varepsilon = Q_\varepsilon u^\varepsilon + s_*^\varepsilon(Q_\varepsilon u^\varepsilon)\}, \quad \varepsilon \in [0, \varepsilon_0],$$

where $s_*^\varepsilon : Y_\varepsilon \rightarrow Z_\varepsilon$ is a Lipschitz continuous map satisfying

$$(15) \quad \|s_*^\varepsilon - s_*^0\| = \sup_{v \in Y_0} \|s_*^\varepsilon(v) - s_*^0(v)\|_{X_\varepsilon^{\frac{1}{2}}} \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}},$$

for some constant C independent of ε . The invariant manifold \mathcal{M}_ε is exponentially attracting and the global attractor \mathcal{A}_ε of the problem (4) lies in \mathcal{M}_ε . The flow on \mathcal{A}_ε is given by

$$u^\varepsilon(t) = v^\varepsilon(t) + s_*^\varepsilon(v^\varepsilon(t)), \quad t \in \mathbb{R},$$

where $v^\varepsilon(t)$ satisfies

$$\dot{v}^\varepsilon + A_\varepsilon^+ v^\varepsilon = Q_\varepsilon f(v^\varepsilon + s_*^\varepsilon(v^\varepsilon(t))).$$

Proof. Given $L, \Delta > 0$ we consider the set

$$\Sigma_\varepsilon = \left\{ s^\varepsilon : Y_\varepsilon \rightarrow Z_\varepsilon; \|s^\varepsilon\| \leq D \text{ and } \|s^\varepsilon(v) - s^\varepsilon(\tilde{v})\|_{X_\varepsilon^{\frac{1}{2}}} \leq \Delta \|v - \tilde{v}\|_{X_\varepsilon^{\frac{1}{2}}} \right\}.$$

It's not difficult to see that $(\Sigma_\varepsilon, \|\cdot\|)$ is a complete metric space. We write the solution u^ε of (4) as $u^\varepsilon = v^\varepsilon + z^\varepsilon$, with $v^\varepsilon \in Y_\varepsilon$ and $z^\varepsilon \in Z_\varepsilon$ and since Q_ε and $I - Q_\varepsilon$ commute with A_ε , we can write

$$(16) \quad \begin{cases} \dot{v}_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = Q_\varepsilon f(v^\varepsilon + z^\varepsilon) := H_\varepsilon(v^\varepsilon, z^\varepsilon) \\ \dot{z}_t^\varepsilon + A_\varepsilon^- z^\varepsilon = (I - Q_\varepsilon) f(v^\varepsilon + z^\varepsilon) := G_\varepsilon(v^\varepsilon, z^\varepsilon). \end{cases}$$

By assumption there is a certain $\rho > 0$ such that for all $v^\varepsilon, \tilde{v}^\varepsilon \in Y_\varepsilon$ and $z^\varepsilon, \tilde{z}^\varepsilon \in Z_\varepsilon$,

$$\begin{aligned} \|H_\varepsilon(v^\varepsilon, z^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \rho, \quad \|G_\varepsilon(v^\varepsilon, z^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \rho, \\ \|H_\varepsilon(v^\varepsilon, z^\varepsilon) - H_\varepsilon(\tilde{v}^\varepsilon, \tilde{z}^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \rho(\|v^\varepsilon - \tilde{v}^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} + \|z^\varepsilon - \tilde{z}^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}}), \\ \|G_\varepsilon(v^\varepsilon, z^\varepsilon) - G_\varepsilon(\tilde{v}^\varepsilon, \tilde{z}^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \rho(\|v^\varepsilon - \tilde{v}^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}} + \|z^\varepsilon - \tilde{z}^\varepsilon\|_{X_\varepsilon^{\frac{1}{2}}}). \end{aligned}$$

Also, for ε sufficiently small, we can choose ρ such that

$$\begin{aligned} \rho M \gamma^{-1} \leq D, \quad 0 \leq \gamma - \beta - \rho M(1 + \Delta), \quad \frac{\rho M^2(1 + \Delta)}{\gamma - \beta - \rho M(1 + \Delta)} \leq \Delta, \\ \rho M \gamma^{-1} + \frac{\rho^2 M^2(1 + \Delta)\beta^{-1}}{\gamma - \beta - \rho M(1 + \Delta)} \leq \frac{1}{2}, \quad L = \left[\rho M + \frac{\rho^2 M^2(1 + \Delta)(1 + M)}{\gamma - \beta - \rho M(1 + \Delta)} \right], \quad \gamma - L > 0. \end{aligned}$$

We will divide the proof in three parts.

Part 1(Existence) Let $s^\varepsilon \in \Sigma_\varepsilon$ and $v^\varepsilon(t) = v^\varepsilon(t, \tau, \eta, s^\varepsilon)$ be the solution of

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = H_\varepsilon(v^\varepsilon, s^\varepsilon(v^\varepsilon)), & t < \tau \\ v^\varepsilon(\tau) = \eta. \end{cases}$$

We define $\Phi_\varepsilon : \Sigma_\varepsilon \rightarrow \Sigma_\varepsilon$ by

$$\Phi_\varepsilon(s^\varepsilon)(\eta) = \int_{-\infty}^{\tau} e^{-A_\varepsilon^-(\tau-r)} G_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r))) dr.$$

Then

$$\|\Phi_\varepsilon(s^\varepsilon)(\eta)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \rho M \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} dr = \rho M \gamma^{-1} \leq D.$$

For $s^\varepsilon, \tilde{s}^\varepsilon \in \Sigma_\varepsilon$, $\eta, \tilde{\eta} \in Y_\varepsilon$, $v^\varepsilon(t) = v^\varepsilon(t, \tau, \eta, s^\varepsilon)$ and $\tilde{v}^\varepsilon(t) = \tilde{v}^\varepsilon(t, \tau, \tilde{\eta}, \tilde{s}^\varepsilon)$ we have

$$\begin{aligned} v^\varepsilon(t) - \tilde{v}^\varepsilon(t) &= e^{-A_\varepsilon^+(t-\tau)}(\eta - \tilde{\eta}) \\ &\quad + \int_{\tau}^t e^{-A_\varepsilon^+(t-r)} [H_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r))) - H_\varepsilon(\tilde{v}^\varepsilon(r), \tilde{s}^\varepsilon(\tilde{v}^\varepsilon(r)))] dr. \end{aligned}$$

Thus,

$$\begin{aligned} \|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq M e^{-\beta(t-\tau)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + M \int_t^{\tau} e^{-\beta(t-r)} \|H_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r))) - H_\varepsilon(\tilde{v}^\varepsilon(r), \tilde{s}^\varepsilon(\tilde{v}^\varepsilon(r)))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\leq M e^{-\beta(t-\tau)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \rho M \int_t^{\tau} e^{-\beta(t-r)} [\|v^\varepsilon(r) - \tilde{v}^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} + \|s^\varepsilon(v^\varepsilon(r)) - \tilde{s}^\varepsilon(\tilde{v}^\varepsilon(r))\|_{X_\varepsilon^{\frac{1}{2}}}] dr \\ &\leq M e^{-\beta(t-\tau)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \rho M \int_t^{\tau} e^{-\beta(t-r)} [(1 + \Delta) \|v^\varepsilon(r) - \tilde{v}^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} + \|s^\varepsilon(\tilde{v}^\varepsilon(r)) - \tilde{s}^\varepsilon(\tilde{v}^\varepsilon(r))\|_{X_\varepsilon^{\frac{1}{2}}}] dr \\ &\leq M e^{-\beta(t-\tau)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \rho M(1 + \Delta) \int_t^{\tau} e^{-\beta(t-r)} \|v^\varepsilon(r) - \tilde{v}^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} dr + \rho M \|s^\varepsilon - \tilde{s}^\varepsilon\| \int_t^{\tau} e^{-\beta(t-r)} dr \\ &\leq M e^{-\beta(t-\tau)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\quad + \rho M(1 + \Delta) \int_t^{\tau} e^{-\beta(t-r)} \|v^\varepsilon(r) - \tilde{v}^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} dr + \rho M \beta^{-1} \|s^\varepsilon - \tilde{s}^\varepsilon\| e^{-\beta(t-\tau)}. \end{aligned}$$

By Gronwall's inequality,

$$\|v^\varepsilon(t) - \tilde{v}^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \left[M\|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} + \rho M\beta^{-1} \|s^\varepsilon - \tilde{s}^\varepsilon\| \right] e^{[\rho M(1+\Delta)+\beta](\tau-t)}.$$

Thus

$$\begin{aligned} \|\Phi_\varepsilon(s^\varepsilon)(\eta) - \Phi_\varepsilon(\tilde{s}^\varepsilon)(\tilde{\eta})\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \int_{-\infty}^{\tau} \|e^{-\tilde{A}_\varepsilon^-(\tau-r)}[G_\varepsilon(v^\varepsilon(r), s^\varepsilon(v^\varepsilon(r))) - G_\varepsilon(\tilde{v}^\varepsilon(r), \tilde{s}^\varepsilon(\tilde{v}^\varepsilon(r)))]\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\leq \frac{\rho M^2(1+\Delta)}{\gamma - \beta - \rho M(1+\Delta)} \|\eta - \tilde{\eta}\|_{X_\varepsilon^{\frac{1}{2}}} + \frac{\rho^2 M^2(1+\Delta)\beta^{-1}}{\gamma - \beta - \rho M(1+\Delta)} \|s^\varepsilon - \tilde{s}^\varepsilon\|. \end{aligned}$$

Therefore Φ_ε is a contraction on Σ_ε , hence there is a unique $s_*^\varepsilon \in \Sigma_\varepsilon$ which is a fixed point of Φ_ε .

Now, let $(\bar{v}^\varepsilon, \bar{z}^\varepsilon) \in \mathcal{M}_\varepsilon$, $\bar{z}^\varepsilon = s_*^\varepsilon(\bar{v}^\varepsilon)$ and let $v_{s_*}^\varepsilon(t)$ be the solution of

$$\begin{cases} v_t^\varepsilon + A_\varepsilon^+ v^\varepsilon = H_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)), & t < \tau \\ v^\varepsilon(0) = \bar{v}^\varepsilon. \end{cases}$$

Thus, $\{(v_{s_*}^\varepsilon(t), s_*^\varepsilon(v_{s_*}^\varepsilon(t)))\}_{t \in \mathbb{R}}$ defines a curve on \mathcal{M}_ε . But the only solution of equation

$$z_t^\varepsilon + A_\varepsilon^- z^\varepsilon = G_\varepsilon(v_{s_*}^\varepsilon(t), s_*^\varepsilon(v_{s_*}^\varepsilon(t)))$$

which stays bounded when $t \rightarrow -\infty$ is given by

$$z_{s_*}^\varepsilon = \int_{-\infty}^t e^{-A_\varepsilon^-(t-r)} G_\varepsilon(v_{s_*}^\varepsilon(r), s_*^\varepsilon(v_{s_*}^\varepsilon(r))) dr = s_*^\varepsilon(v_{s_*}^\varepsilon(t)).$$

Therefore $(v_{s_*}^\varepsilon(t), s_*^\varepsilon(v_{s_*}^\varepsilon(t)))$ is a solution of (16) through $(\bar{v}^\varepsilon, \bar{z}^\varepsilon)$ and thus \mathcal{M}_ε is a invariant manifold for (4).

Part 2(Estimate) Now we will prove the estimate (15). For $\eta \in Y_0$, we have

$$\begin{aligned} \|s_*^\varepsilon(\eta) - s_*^0(\eta)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \int_{-\infty}^{\tau} \|e^{-A_\varepsilon^-(\tau-r)} G_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)) - e^{-A_0^-(\tau-r)} G_0(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\leq \int_{-\infty}^{\tau} \|e^{-A_\varepsilon^-(\tau-r)} G_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)) - e^{-A_\varepsilon^-(\tau-r)} G_\varepsilon(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\quad + \int_{-\infty}^{\tau} \|e^{-A_\varepsilon^-(\tau-r)} G_\varepsilon(v^0, s_*^0(v^0)) - e^{-A_\varepsilon^-(\tau-r)} G_0(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\quad + \int_{-\infty}^{\tau} \|e^{-A_\varepsilon^-(\tau-r)} G_0(v^0, s_*^0(v^0)) - e^{-A_0^-(\tau-r)} G_0(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr. \end{aligned}$$

If we denote the last three integrals for I_1 , I_2 and I_3 respectively, we have

$$\begin{aligned} I_1 &\leq \rho M \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} [(1+\Delta) \|v^\varepsilon - v^0\|_{X_\varepsilon^{\frac{1}{2}}} + \|s_*^\varepsilon - s_*^0\|] dr \\ &\leq \rho M(1+\Delta) \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} \|v^\varepsilon - v^0\|_{X_\varepsilon^{\frac{1}{2}}} dr + \rho M \|s_*^\varepsilon - s_*^0\| \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} dr \\ &= \rho M \gamma^{-1} \|s_*^\varepsilon - s_*^0\| + \rho M(1+\Delta) \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} \|v^\varepsilon - v^0\|_{X_\varepsilon^{\frac{1}{2}}} dr. \end{aligned}$$

For I_2 we have

$$G_\varepsilon(v^0, s_*^0(v^0)) - G_0(v^0, s_*^0(v^0)) = (Q_\varepsilon - Q_0)f(v^0 + s_*^0(v^0)),$$

and if we denote $\tau(\varepsilon) = (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}$, then $I_2 \leq C\tau(\varepsilon)$. And for I_3 , we have

$$I_3 \leq \int_{-\infty}^{\tau} l_\varepsilon(\tau - r)e^{-\bar{\gamma}(\tau-r)} dr \leq C\tau(\varepsilon)|\log(\tau(\varepsilon))|,$$

where we have used the Lemma 3.10 in [16]. Thus

$$\begin{aligned} \|s_*^\varepsilon(\eta) - s_*^0(\eta)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq C\tau(\varepsilon)|\log(\tau(\varepsilon))| \\ &\quad + \rho M\gamma^{-1}\|s_*^\varepsilon - s_*^0\| + \rho M(1 + \Delta) \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} \|v^\varepsilon - v^0\|_{X_\varepsilon^{\frac{1}{2}}} dr. \end{aligned}$$

But,

$$\begin{aligned} \|v^\varepsilon(t) - v^0(t)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \|(e^{-A_\varepsilon^+(t-\tau)} - e^{-A_0^+(t-\tau)})\eta\| \\ &\quad + \int_t^\tau \|e^{-A_\varepsilon^+(t-r)} H_\varepsilon(v^\varepsilon, s_*^\varepsilon(v^\varepsilon)) - e^{-A_\varepsilon^+(t-r)} H_\varepsilon(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\quad + \int_t^\tau \|e^{-A_\varepsilon^+(t-r)} H_\varepsilon(v^0, s_*^0(v^0)) - e^{-A_\varepsilon^+(t-r)} H_0(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \\ &\quad + \int_t^\tau \|e^{-A_\varepsilon^+(t-r)} H_0(v^0, s_*^0(v^0)) - e^{-A_0^+(t-r)} H_0(v^0, s_*^0(v^0))\|_{X_\varepsilon^{\frac{1}{2}}} dr \end{aligned}$$

With the same argument used earlier, we have

$$\begin{aligned} \|v^\varepsilon(t) - v^0(t)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq C\tau(\varepsilon) \int_t^\tau e^{-\beta(t-r)} dr \\ &\quad + \rho M\|s_*^\varepsilon - s_*^0\| \int_t^\tau e^{-\beta(t-r)} dr + \rho M(1 + \Delta) \int_t^\tau e^{-\beta(t-r)} \|v^\varepsilon - v^0\|_{X_\varepsilon^{\frac{1}{2}}} dr. \end{aligned}$$

By Gronwall's inequality,

$$\|v^\varepsilon(t) - v^0(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq [C\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho M\beta^{-1}\|s_*^\varepsilon - s_*^0\|]e^{[\rho M(1+\Delta)+\beta](\tau-t)},$$

thus

$$\begin{aligned} \|s_*^\varepsilon(\eta) - s_*^0(\eta)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq C\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho M\gamma^{-1}\|s_*^\varepsilon - s_*^0\| \\ &\quad + \rho M(1 + \Delta) \int_{-\infty}^{\tau} e^{-\gamma(\tau-r)} e^{[\rho M(1+\Delta)+\beta](\tau-r)} dr [C\tau(\varepsilon)|\log(\tau(\varepsilon))| + \rho M\beta^{-1}\|s_*^\varepsilon - s_*^0\|] \\ &\leq C\tau(\varepsilon)|\log(\tau(\varepsilon))| + \left[\rho M\gamma^{-1} + \frac{\rho^2 M^2(1 + \Delta)\beta^{-1}}{\gamma - \beta - \rho M(1 + \Delta)} \right] \|s_*^\varepsilon - s_*^0\| \end{aligned}$$

which implies $\|s_*^\varepsilon - s_*^0\| \leq C\tau(\varepsilon)|\log(\tau(\varepsilon))|$.

Part 3(Exponential attraction) It remains shown that \mathcal{M}_ε is exponentially attracting and $\mathcal{A}_\varepsilon \subset \mathcal{M}_\varepsilon$. Let $(v^\varepsilon, z^\varepsilon) \in Y_\varepsilon \oplus Z_\varepsilon$ be the solution of (16) and define $\xi^\varepsilon(t) = z^\varepsilon - s_*^\varepsilon(v^\varepsilon(t))$ and consider $y^\varepsilon(r, t), r \leq t, t \geq 0$, the solution of

$$\begin{cases} y_t^\varepsilon + A_\varepsilon^+ y^\varepsilon = H_\varepsilon(y^\varepsilon, s_*^\varepsilon(y^\varepsilon)), & r \leq t \\ y^\varepsilon(t, t) = v^\varepsilon(t). \end{cases}$$

Thus,

$$\begin{aligned} & \|y^\varepsilon(r, t) - v^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} \\ &= \left\| \int_t^r e^{-A_\varepsilon^+(r-\theta)} [H_\varepsilon(y^\varepsilon(\theta, t), s_*^\varepsilon(y^\varepsilon(\theta, t))) - H_\varepsilon(v^\varepsilon(\theta), z^\varepsilon(\theta))] d\theta \right\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq \rho M \int_r^t e^{-\beta(r-\theta)} [(1 + \Delta) \|y^\varepsilon(\theta, t) - v^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} + \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}}] d\theta. \end{aligned}$$

By Gronwall's inequality

$$\|y^\varepsilon(r, t) - v^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \rho M \int_r^t e^{-(\beta - \rho M(1+\Delta))(\theta-r)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta \quad r \leq t.$$

Now we take $t_0 \in [r, t]$ and then

$$\begin{aligned} & \|y^\varepsilon(r, t) - y^\varepsilon(r, t_0)\|_{X_\varepsilon^{\frac{1}{2}}} \\ &= \|e^{-A_\varepsilon^+(r-t_0)} [y(t_0, t) - v^\varepsilon(t_0)]\|_{X_\varepsilon^{\frac{1}{2}}} \\ &+ \left\| \int_{t_0}^r e^{-A_\varepsilon^+(r-\theta)} [H_\varepsilon(y^\varepsilon(\theta, t), s_*^\varepsilon(y^\varepsilon(\theta, t))) - H_\varepsilon(y^\varepsilon(\theta, t_0), s_*^\varepsilon(y^\varepsilon(\theta, t_0)))] d\theta \right\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq \rho M^2 e^{-\beta(r-t_0)} \int_{t_0}^t e^{-(\beta - \rho M(1+\Delta))(\theta-t_0)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta \\ &+ \rho M \int_r^{t_0} e^{-\beta(r-\theta)} (1 + \Delta) \|y^\varepsilon(\theta, t) - y^\varepsilon(\theta, t_0)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta. \end{aligned}$$

By Gronwall's inequality

$$\|y^\varepsilon(r, t) - y^\varepsilon(r, t_0)\|_{X_\varepsilon^{\frac{1}{2}}} \leq \rho M^2 \int_{t_0}^t e^{-(\beta - \rho M(1+\Delta))(\theta-r)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta.$$

In what follows we estimate $\xi^\varepsilon(t)$. Since

$$z^\varepsilon(t) = e^{-A_\varepsilon^-(t-t_0)} z^\varepsilon(t_0) + \int_{t_0}^t e^{-A_\varepsilon^-(t-r)} G_\varepsilon(v^\varepsilon(r), z^\varepsilon(r)) dr,$$

we have

$$\begin{aligned}
\xi^\varepsilon(t) - e^{-A_\varepsilon^-(t-t_0)}\xi^\varepsilon(t_0) &= z^\varepsilon(t) - s_*^\varepsilon(v^\varepsilon(t)) - e^{-A_\varepsilon^-(t-t_0)}[z^\varepsilon(t_0) - s_*^\varepsilon(v^\varepsilon(t_0))] \\
&= \int_{t_0}^t e^{-A_\varepsilon^-(t-r)} G_\varepsilon(v^\varepsilon(r), z^\varepsilon(r)) dr - s_*^\varepsilon(v^\varepsilon(t)) + e^{-A_\varepsilon^-(t-t_0)} s_*^\varepsilon(v^\varepsilon(t_0)) \\
&= \int_{t_0}^t e^{-A_\varepsilon^-(t-r)} G_\varepsilon(v^\varepsilon(r), z^\varepsilon(r)) dr - \int_{-\infty}^t e^{-A_\varepsilon^-(t-r)} G_\varepsilon(y^\varepsilon(r, t), s_*^\varepsilon(y^\varepsilon(r, t))) dr \\
&\quad + e^{-A_\varepsilon^-(t-t_0)} \int_{-\infty}^{t_0} e^{-A_\varepsilon^-(t_0-r)} G_\varepsilon(y^\varepsilon(r, t_0), s_*^\varepsilon(y^\varepsilon(r, t_0))) dr \\
&= \int_{t_0}^t e^{-A_\varepsilon^-(t-r)} [G_\varepsilon(v^\varepsilon(r), z^\varepsilon(r)) - G_\varepsilon(y^\varepsilon(r, t), s_*^\varepsilon(y^\varepsilon(r, t)))] dr \\
&\quad - \int_{-\infty}^{t_0} e^{-A_\varepsilon^-(t-r)} [G_\varepsilon(y^\varepsilon(r, t), s_*^\varepsilon(y^\varepsilon(r, t))) - G_\varepsilon(y^\varepsilon(r, t_0), s_*^\varepsilon(y^\varepsilon(r, t_0)))] dr.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\|\xi^\varepsilon(t) - e^{-A_\varepsilon^-(t-t_0)}\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} \\
&\leq \rho M \int_{t_0}^t e^{-\gamma(t-r)} [\|v^\varepsilon(r) - y^\varepsilon(r, t)\|_{X_\varepsilon^{\frac{1}{2}}} + \|z^\varepsilon(r) - s_*^\varepsilon(y^\varepsilon(r, t))\|_{X_\varepsilon^{\frac{1}{2}}}] dr \\
&\quad + \rho M(1 + \Delta) \int_{-\infty}^{t_0} e^{-\gamma(t-r)} \|y^\varepsilon(r, t) - y^\varepsilon(r, t_0)\|_{X_\varepsilon^{\frac{1}{2}}} dr \\
&\leq \rho M \int_{t_0}^t e^{-\gamma(t-r)} \|\xi^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} dr \\
&\quad + \rho^2 M^2(1 + \Delta) e^{-\gamma t} \int_{t_0}^t e^{-(\gamma - \beta - \rho M(1 + \Delta))\theta} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} \int_{-\infty}^\theta e^{(\gamma - \beta - \rho M(1 + \Delta))r} dr d\theta \\
&\quad + \rho^2 M^3(1 + \Delta) e^{-\gamma t} \int_{t_0}^t e^{-(\gamma - \beta - \rho M(1 + \Delta))\theta} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} \int_{-\infty}^{t_0} e^{(\beta - \gamma - \rho M(1 + \Delta))r} dr d\theta,
\end{aligned}$$

and then

$$\begin{aligned}
\|\xi^\varepsilon(t) - e^{-A_\varepsilon^-(t-t_0)}\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq [\rho M - \frac{\rho^2 M^2(1 + \Delta)}{\gamma - \beta - \rho M(1 + \Delta)}] \int_{t_0}^t e^{-\gamma(t-\theta)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta \\
&\quad + \frac{\rho^2 M^3(1 + \Delta)}{\gamma - \beta - \rho M(1 + \Delta)} \int_{t_0}^t e^{-(\gamma - \beta - \rho M(1 + \Delta))(\theta - t_0)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta.
\end{aligned}$$

Hence

$$\begin{aligned}
e^{\gamma(t-t_0)} \|\xi^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} &\leq M \|\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} + \left[\rho M + \frac{\rho^2 M^2(1 + \Delta)}{\gamma - \beta - \rho M(1 + \Delta)} \right] \int_{t_0}^t e^{\gamma(r-t_0)} \|\xi^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} dr \\
&\quad + \frac{\rho^2 M^3(1 + \Delta)}{\gamma - \beta - \rho M(1 + \Delta)} \int_{t_0}^t e^{-(\gamma - \beta - \rho M(1 + \Delta))(\theta - t_0)} e^{\beta(\theta - t_0)} \|\xi^\varepsilon(\theta)\|_{X_\varepsilon^{\frac{1}{2}}} d\theta \\
&\leq M \|\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} + \left[\rho M + \frac{\rho^2 M^2(1 + \Delta)(1 + M)}{\gamma - \beta - \rho M(1 + \Delta)} \right] \int_{t_0}^t e^{\gamma(r-t_0)} \|\xi^\varepsilon(r)\|_{X_\varepsilon^{\frac{1}{2}}} dr.
\end{aligned}$$

By Gronwall's inequality

$$\|\xi^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq M \|\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} e^{-(\gamma-L)(t-t_0)},$$

and then

$$\|z^\varepsilon(t) - s_*^\varepsilon(v^\varepsilon(t))\|_{X_\varepsilon^{\frac{1}{2}}} = \|\xi^\varepsilon(t)\|_{X_\varepsilon^{\frac{1}{2}}} \leq M \|\xi^\varepsilon(t_0)\|_{X_\varepsilon^{\frac{1}{2}}} e^{-(\gamma-L)(t-t_0)}.$$

Now if $u^\varepsilon := T_\varepsilon(t)u_0^\varepsilon = v^\varepsilon(t) + z^\varepsilon(t)$, $t \in \mathbb{R}$, denotes the solution through at $u_0^\varepsilon = v_0^\varepsilon + z_0^\varepsilon \in \mathcal{A}_\varepsilon$, then

$$\|z^\varepsilon(t) - s_*^\varepsilon(v^\varepsilon(t))\|_{X_\varepsilon^{\frac{1}{2}}} \leq M \|z_0^\varepsilon - s_*^\varepsilon(v_0^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}} e^{-(\gamma-L)(t-t_0)}.$$

Since $\{T_\varepsilon(t)u_0^\varepsilon; t \in \mathbb{R}\} \subset \mathcal{A}_\varepsilon$ is bounded, letting $t_0 \rightarrow -\infty$ we obtain $T_\varepsilon(t)u_0^\varepsilon = v^\varepsilon(t) + s_*^\varepsilon(v^\varepsilon(t)) \in \mathcal{M}_\varepsilon$. That is $\mathcal{A}_\varepsilon \subset \mathcal{M}_\varepsilon$. Moreover, if $B_\varepsilon \subset X_\varepsilon^{\frac{1}{2}}$ is a bounded set and $u_0^\varepsilon = v_0^\varepsilon + z_0^\varepsilon \in B_\varepsilon$, we conclude that $T_\varepsilon(t)u_0^\varepsilon = v^\varepsilon(t) + z^\varepsilon(t)$ satisfies

$$\begin{aligned} \sup_{u_0^\varepsilon \in B_\varepsilon} \inf_{w \in \mathcal{M}_\varepsilon} \|T_\varepsilon(t)u_0^\varepsilon - w\|_{X_\varepsilon^{\frac{1}{2}}} &\leq \sup_{u_0^\varepsilon \in B_\varepsilon} \|z^\varepsilon(t) - s_*^\varepsilon(v^\varepsilon(t))\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq M e^{-(\gamma-L)(t-t_0)} \sup_{u_0^\varepsilon \in B_\varepsilon} \|z_0^\varepsilon - s_*^\varepsilon(v_0^\varepsilon)\|_{X_\varepsilon^{\frac{1}{2}}}, \end{aligned}$$

which implies

$$\text{dist}_H(T_\varepsilon(t)B_\varepsilon, \mathcal{M}_\varepsilon) \leq C(B_\varepsilon) e^{-(\gamma-L)(t-t_0)},$$

and thus the proof is complete. \square

Remark 4.2. It is well known the C^0 , C^1 and $C^{1,\theta}$ convergences of invariant manifolds (see [7] and [16]). That is $\|s_*^\varepsilon - s_*^0\|_{C^0(Y_0)}, \|s_*^\varepsilon - s_*^0\|_{C^1(Y_0)}, \|s_*^\varepsilon - s_*^0\|_{C^{1,\theta}(Y_0)} \xrightarrow{\varepsilon \rightarrow 0} 0$.

5. RATE OF CONVERGENCE OF ATTRACTORS

In this section we will estimate the continuity of attractors of (4) in the Hausdorff metric by the rate of convergence of resolvent operators obtained in the Section 3.

The operator A_ε , $\varepsilon \in [0, \varepsilon_0]$, has compact resolvent and according to [7], A_0 is Sturm-Liouville type, which implies transversality of stable and unstable manifolds of the equilibrium points. Since we assume hyperbolicity, the limiting problem (3) generates a Morse-Smale semigroup in $X_0^{\frac{1}{2}}$ and hence the perturbed problem (1) generates a Morse-Smale semigroup in $X_\varepsilon^{\frac{1}{2}}$.

We saw in the last section how the gap condition implies the existence of the finite dimensional invariant manifold \mathcal{M}_ε for (4). The invariant manifold contains the attractor \mathcal{A}_ε and the flow is given by an ordinary differential equation. That is,

$$u^\varepsilon(t) = v^\varepsilon(t) + s_*^\varepsilon(v^\varepsilon(t)), \quad t \in \mathbb{R},$$

where $v^\varepsilon(t)$ satisfies

$$\dot{v}^\varepsilon + A_\varepsilon^+ v^\varepsilon = Q_\varepsilon f(v^\varepsilon + s_*^\varepsilon(v^\varepsilon(t))),$$

and we can consider $v^\varepsilon \in \mathbb{R}^m$ and $H_\varepsilon(v^\varepsilon) = Q_\varepsilon f(v^\varepsilon + s_*^\varepsilon(v^\varepsilon(t)))$ a continuously differentiable map in \mathbb{R}^m . For each $\varepsilon \in [0, \varepsilon_0]$, we denote $\tilde{T}_\varepsilon = T_\varepsilon(1)|_{\mathcal{M}_\varepsilon}$ the time one map of the nonlinear semigroup $T_\varepsilon(\cdot)$ restricted to the invariant manifold \mathcal{M}_ε . By Remark 4.2 and since $Q_\varepsilon \rightarrow Q_0$ as $\varepsilon \rightarrow 0$ in \mathbb{R}^m , we have

$$(17) \quad \|\tilde{T}_\varepsilon - \tilde{T}_0\|_{C^1(\mathbb{R}^m, \mathbb{R}^m)} \xrightarrow{\varepsilon \rightarrow 0} 0 \quad \text{and} \quad \|\tilde{T}_\varepsilon - \tilde{T}_0\|_{L^\infty(\mathbb{R}^m, \mathbb{R}^m)} \leq C\tau(\varepsilon)|\log(\tau(\varepsilon))|,$$

where $\tau(\varepsilon) = (\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}}$ and the last estimate was proved as Theorem 2.3.

Since we have a Morse-Smale semigroup in \mathbb{R}^m , by using techniques of shadowing in [14] and [16] we have the following result.

Proposition 5.1. Let $T : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a discrete Morse-Smale semigroup with a global attractor \mathcal{A} . Then there are a positive constant L , a neighborhood $\mathcal{N}(\mathcal{A})$ of \mathcal{A} and a neighborhood Θ of T in the $C^1(\mathcal{N}(\mathcal{A}), \mathbb{R}^m)$ topology such that, for any $T_1, T_2 \in \Theta$ with attractors $\mathcal{A}_1, \mathcal{A}_2$ respectively attractors, we have

$$\text{dist}_H(\mathcal{A}_1, \mathcal{A}_2) \leq L\|T_1 - T_2\|_{L^\infty(\mathcal{N}(\mathcal{A}), \mathbb{R}^m)}.$$

Theorem 5.2. Let \mathcal{A}_ε , $\varepsilon \in [0, \varepsilon_0]$, be the attractor for (4). Then there is a positive constant C independent of ε such that

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} |\log(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)|.$$

Proof. We will follow [16]. For each $\varepsilon \in [0, \varepsilon_0]$ we denote $T_\varepsilon = T_\varepsilon(1)$. Given $u^\varepsilon \in \mathcal{A}_\varepsilon$, by invariance there is $w^\varepsilon \in \mathcal{A}_\varepsilon$ such that $u^\varepsilon = T_\varepsilon w^\varepsilon$ so we can write $w^\varepsilon = Q_\varepsilon w^\varepsilon + s_*^\varepsilon(Q_\varepsilon w^\varepsilon)$ where $Q_\varepsilon w^\varepsilon \in \bar{\mathcal{A}}_\varepsilon$ the projected attractor in \mathbb{R}^m ($\bar{\mathcal{A}}_\varepsilon = Q_\varepsilon \mathcal{A}_\varepsilon$). Thus

$$\begin{aligned} \|u^\varepsilon - u^0\|_{X_\varepsilon^{\frac{1}{2}}} &= \|T_\varepsilon w^\varepsilon - T_0 w^0\|_{X_\varepsilon^{\frac{1}{2}}} \leq \|T_\varepsilon w^\varepsilon - T_\varepsilon w^0\|_{X_\varepsilon^{\frac{1}{2}}} + \|T_\varepsilon w^0 - T_0 w^0\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} |\log(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)| + C\|w^\varepsilon - w^0\|. \end{aligned}$$

But

$$\begin{aligned} \|w^\varepsilon - w^0\| &= \|Q_\varepsilon w^\varepsilon - Q_0 w^0\|_{\mathbb{R}^m} + \|s_*^\varepsilon(Q_\varepsilon w^\varepsilon) - s_*^0(Q_0 w^0)\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq \|Q_\varepsilon w^\varepsilon - Q_0 w^0\|_{\mathbb{R}^m} + \|s_*^\varepsilon(Q_\varepsilon w^\varepsilon) - s_*^\varepsilon(Q_0 w^0)\|_{X_\varepsilon^{\frac{1}{2}}} + \|s_*^\varepsilon(Q_0 w^0) - s_*^0(Q_0 w^0)\|_{X_\varepsilon^{\frac{1}{2}}} \\ &\leq C\|Q_\varepsilon w^\varepsilon - Q_0 w^0\|_{\mathbb{R}^m} + C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} |\log(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)|, \end{aligned}$$

which implies

$$d_H(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leq d_H(\bar{\mathcal{A}}_\varepsilon, \bar{\mathcal{A}}_0) + C(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)^{\frac{1}{2}} |\log(\|p_\varepsilon - p_0\|_{L^\infty(\Omega_1)} + \varepsilon)|.$$

The result follows by (17) and Proposition 5.1. \square

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